3 The integral

Start with simple functions, $f : X \to [-\infty, \infty]$ is simple if it takes only finitely many values.

Definition. If f is a non-negative (initially, just to be careful about cancellation), simple, measurable functions on $(X, \mathcal{F}, \mu)$ and $E \in \mathcal{F}$, then

$$I_E(f) = \sum_{c_i \in (0,\infty]} c_i \mu(f^{-1}(c_i) \cap E)$$

Note that its a finite sum, and that the integral could be $+\infty$. Since f is measurable, $f^{-1}(c_i) \in \mathcal{F}$ and $I_E(f)$ is well defined.

Properties of the integral.

Theorem. *If $s_i : X \to \mathbb{R}$ are simple and measurable then*

1. $I_E(cs_i) = cI_E(s_i), c \geq 0$

2. $I_E(s_1 + s_2) = I_E(s_1) + I_E(s_2)$

3. $s_1 \le s_2 \Rightarrow I_E(s_1) \le I_E(s_2)$.

Proof. We just prove the second one. Let $E_i = s_1^{-1}(c_i)$ (the sets on which s_1 is constant, and $F_i = s_2^{-1}(c_i)$ (the sets on which s_2 is constant. Then $X = \bigcup_i E_i = \bigcup_i F_i$, and on $E_i \cap F_i$, $s_1 + s_2$ is constant. So

$$\begin{aligned} I_E(s_1+s_2) = \sum_{i,j}(c_i + c_j')\mu(E_i \cap F_j \cap E) &= \sum_{d\in(0,\infty)} d\mu((s_1+s_2)^{-1}(d) \cap E) \\ &= \sum_{i,j} c_i\mu(E_i \cap F_j \cap E) + \sum_{i,j} c_j'\mu(E_i \cap F_j \cap E) \\ &= \sum_i c_i \sum_j \mu(E_i \cap F_j \cap E) + \sum_j c_j' \sum_i \mu(E_i \cap F_j \cap E) \end{aligned}$$

by $\bigcup_j E_i \cap F_j \cap E = E_i \cap E$, because they are disjoint unions, and so $I_E(s_1+s_2) = I_E(s_1) + I_E(s_2)$. □

For Riemann integrals we forced $f^{-1}(c_i) \cap E$to be intervals, so the Lebesgue integral is very different.

Definition. $f : X \to [0,\infty]$ is a non-negative and measurable function, then we define

$$\int_E f\ d\mu = \sup\{I_E(s) | 0 \le s \le f \text{ simple, non-negative, measurable}\}$$

We pay for such an easy definition. We need to show that there exist such simple functions

Proposition. *If f is measurable, $\exists$ a sequence of simple measurable functions $0 \le s_i \le s_{i+1} \le f$, such that $s_i(x) \to f(x)$ and on any subset of X on which f is bounded, $s_i(x) \to f(x)$ uniformly*

Proof. Suppose we take $[0, n) \subset \mathbb{R}$. Divide this into $n2^n$ subintervals of length 2^n.

$$I_i = \left\{ t \in \mathbb{R} \middle| \frac{i-1}{2^n} \le t \le \frac{i}{2^n} \right\}, \qquad 1 \le i \le n2^n$$

Then we define the sets $E_i = f^{-1}(I_i)$ and $F_n = f^{-1}([n,\infty])$. Essentially, each E_i is the set on which f attains a value between $(i-1)/2^n$ and $i/2^n$, i.e., $x \in E_i \Rightarrow (i-1)/2^n \le f(x) \le i/2^n$. And F_n is the set on which f attains a value in $[n,\infty]$. Then $X = \bigcup_i E_i \cup F_n$. Define

$$s_n(x) = \sum_{i=1}^{n2^n} \left(\frac{i-1}{2^n}\right) \chi_{E_i}(x) + n\chi_{F_n}(x)$$

So on E_i, $s_n(x) \le f(x)$ and on F_n, $n \le f$ and $s_n = n$, so $s_n(x) \le f$.

Now to show that $s_n \le s_{n+1}$. For s_{n+1}, each E_i from s_n gets partitioned into two parts:

$$\left[\frac{i-1}{2^n}, \frac{2i-1}{2^{n+1}}\right), \qquad \left[\frac{2i-1}{2^{n+1}}, \frac{i}{2^n}\right)$$

then s_{n+1} is $(i-1)/2^n$ and $(2i-1)/2^n$ on the same area where s_n is just $(i-1)/2^n$. Further, if we look at $[n, n+1]$, s_{n+1} is n on this interval, as it was for s_n. So we have shown that $s_n \leq f$ and $s_n \leq s_{n+1}$.

Now we need to show that $s_n \to f$. If $f(x) = \infty$, $x \in F_n$ $\forall n$. So $s_n(x) = n$, $\forall n$, so $s_n \to \infty$. For the finite case there is some n_0 such that $f(x) < n_0$. If $f(x) < n_0$, then $\forall n > n_0$ the intervals I_i range from $[0, n]$ at least, so $f(x)$ is in some I_i since $f(x) < n$, so

$$\frac{i-1}{2^n} \leq f(x) \leq \frac{i}{2^n}, \qquad s_n(x) = \frac{i-1}{2^n}$$

so $|f(x) - s_n(x)| < 1/2^n$ for $n > n_0$, so $s_n \to f$.

If f is bounded, then $f(x) < n_0$, $\forall x \in X$, then $|f(x) - s_n(x)| < 1/2^n$, $\forall x \in X$, so $s_n \to f$ uniformly. □

Theorem. *f, g measurable, $E, F \in \mathcal{F}$, then*

1. $f \leq g \Rightarrow \int_E f d\mu \leq \int_E g d\mu$
2. $E \subset F \Rightarrow \int_E f d\mu \leq \int_F f d\mu$
3. $\mu(E) = 0 \Rightarrow \int_E f d\mu = 0$

Proof. 1. Let s be simple, $s \leq f$ then $s \leq g$ so $\sup_{s \leq f} I_E(s) \leq \sup_{s \leq g} I_E(s)$

2. This is true for simple functions,by definitions of the integrals in terms of measures, so true for their supremum.

3. $I_E(s) = \sum c_i \mu(s^{-1}(c_i) \cap E)$ which is 0 by measure theory

□

Theorem. *(Chebychev) $f \geq 0$ measurable, $E \in \mathcal{F}$, $c > 0$ and $E_c = \{x \in E | f(x) \geq c\}$ then*

$$\mu(E_c) \leq \frac{1}{c} \int_E f \, d\mu$$

Proof. Rewrite as $c\mu(E_c) \leq \int_E f d\mu$ Define

$$s = \begin{cases} c, & \text{on } E_c \\ 0, & \text{on } E_c^c \end{cases}$$

so $0 \leq s \leq f$ and

$$I_E(s) \leq \int_E f d\mu, \qquad I_E(s) = c\mu(E_c)$$

□

Lemma. *$f \geq 0$ measurable and $\int_E f d\mu < \infty$, $E \in \mathcal{F}$ then*

$$\mu(\{x \in E, f(x) = \infty\}) = 0$$

Proof.

$$\{x \in E | f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in X | f(x) \geq n\} = \bigcap_{n=1}^{\infty} E_n$$

and

$$\mu(E_n) \leq \frac{1}{n} \int_E f d\mu \to 0 \text{ as } n \to \infty$$

□

Theorem. *If $f \geq 0$, measurable, $\{A_i\}_{i=1}^{\infty}$, $A_i \in \mathcal{F}$, $A_i \cap A_j = \emptyset$, then*

$$\int_E f d\mu = \sum_{i=1}^{\infty} \int_{A_i} f d\mu, \qquad E = \bigcup_{i=1} A_i$$

Proof. As usual, we proceed by proving an inequality in both directions.

Given $\epsilon > 0$, $\exists 0 \leq s \leq f$, s simple, measurable (called a *test function*) such that

$$\int_E f d\mu \leq I_E(s) + \epsilon$$

And we already know additivity for simple functions, $I_E(s) = \sum_{i=1}^{\infty} I_{A_i}(s)$ by the countable additivity of measures. But

$$I_E(s) \leq \sum_{i=1}^{\infty} \int_{A_i} f d\mu \Rightarrow \int_E f d\mu \leq \sum_{i=1}^{\infty} \int_{A_i} f d\mu + \epsilon, \qquad \forall \epsilon$$

Since ϵ is arbitrary, we have the inequality in one direction.

To get the other direction. Let $E = A_1 \cup A_2$, s_1, s_2 simple measurable such that $0 \leq s_i \leq f$ such that $\int_{A_i} s_i d\mu \geq \int_{A_i} f d\mu - \epsilon/2$. But then set $s = \max(s_1, s_2)$, so

$$\int_{A_i} s d\mu \geq \int_{A_i} s_i d\mu \geq \int_{A_i} f d\mu - \epsilon/2$$

But we know for simple functions that

$$\int_E f d\mu \geq \int_E s d\mu = \int_{A_1} s d\mu + \int_{A_2} s d\mu \geq \int_{A_1} f d\mu + \int_{A_2} f d\mu - \epsilon$$

since ϵ is arbitrary, we have established the inequality in the other direction (and the rest of the finite cases follow by induction). So $E = \bigcup_{i=1}^{N} A_i$, $A_i \cap A_k = \emptyset$ then $\int_E f d\mu = \sum_{i=1}^{N} \int_{A_i} f d\mu$.

In general if $E = \bigcup_{i=1}^{\infty} A_i$ then

$$\int_E f d\mu \geq \int_{E_N} f d\mu = \sum_{i=1}^{N} \int_{A_i} f d\mu, \qquad \forall N$$

Since this is true for all N, the infinite inequality follows. □

This is sort of the same as for measures, once we have finite case, we can prove infinite case by hedging it in somehow.

Corollary. *If E_i measurable and $E_1 \subset E_2 \cdots \subset E_N$, $E = \bigcup_{i=1}^{\infty} E_i$ then*

$$\lim_{n\to\infty} \int_{E_n} f d\mu = \int_E f d\mu$$

Proof. Apply the above theorem with $A_1 = E_1, A_2 = E_2 \setminus E_1, A_i = E_i \setminus E_{i-1}$. Then $E = \bigcup_{i=1}^{\infty} A_1$, $A_i \cap A_j = \emptyset$ and

$$\int_E f d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{A_i} f d\mu = \lim_{n\to\infty} \int_{E_n} f d\mu$$

□